The rate equation given in Figure 3.2.11 arises from Bernoulli's principle in fluid dynamics, which states that the quantity $P + \frac{1}{2}\rho v^2 + \rho gh$ is constant. Here P is pressure, ρ is fluid density, v is velocity, and g is the acceleration due to gravity. Comparing the top of the fluid, at the height h, to the fluid at the hole, we have

$$P_{\text{top}} + \tfrac{1}{2}\rho v_{\text{top}}^2 + \rho gh = P_{\text{hole}} + \tfrac{1}{2}\rho v_{\text{hole}}^2 + \rho g \cdot 0.$$

If the pressure at the top and the pressure at the bottom are both atmospheric pressure and if the drainage hole radius is much less than the radius of the bucket, then $P_{\text{top}} = P_{\text{hole}}$ and $v_{\text{top}} = 0$, so $\rho gh = \frac{1}{2}\rho v_{\text{hole}}^2$ leads to Torricelli's law: $v = \sqrt{2gh}$. Since $\dfrac{dV}{dt} = -A_{\text{hole}}v$, we have the differential equation

$$\frac{dV}{dt} = -A_{\text{hole}}\sqrt{2gh}.$$

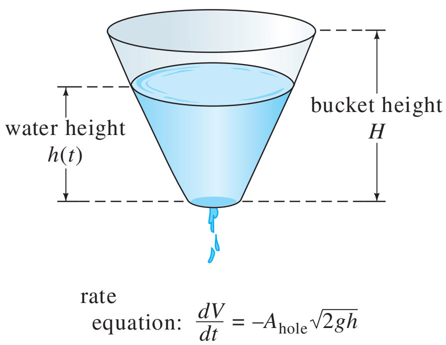

FIGURE 3.2.11 Bucket Drainage

In this problem, we seek a comparison of Torricelli's differential equation with actual data.

(a) If the water is at a height h, we can find the volume of water in the bucket by the formula

$$V(h) = \frac{\pi}{3m}\left[(mh + R_B)^3 - R_B^3\right]$$

in which $m = (R_T - R_B)/H$. Here R_T and R_B denote the top and bottom radii of the bucket, respectively, and H denotes the height of the bucket. Taking this formula as given, differentiate to find a relationship between the rates dV/dt and dh/dt.

(b) Use the relationship derived in part (a) to find a differential equation for $h(t)$ (that is, you should have an independent variable t, a dependent variable h, and constants in the equation).

(c) Solve this differential equation using separation of variables. It is relatively straightforward to determine time as a function of height, but solving for height as a function of time may be difficult.

(d) Obtain a flowerpot, fill it with water, and watch it drain. At a fixed set of heights, record the time at which the water reaches the height. Compare the results to the differential equation's solution.

(e) It has been observed that a more accurate differential equation is

$$\frac{dV}{dt} = -(0.84)A_{\text{hole}}\sqrt{gh}.$$

Solve this differential equation and compare to the results of part (d).

3.3 MODELING WITH SYSTEMS OF FIRST-ORDER DEs

REVIEW MATERIAL

- Section 1.3

INTRODUCTION This section is similar to Section 1.3 in that we are just going to discuss certain mathematical models, but instead of a single differential equation the models will be systems of first-order differential equations. Although some of the models will be based on topics that we explored in the preceding two sections, we are not going to develop any general methods for solving these systems. There are reasons for this: First, we do not possess the necessary mathematical tools for solving systems at this point. Second, some of the systems that we discuss—notably the systems of *nonlinear* first-order DEs—simply cannot be solved analytically. We shall examine solution methods for systems of *linear* DEs in Chapters 4, 7, and 8.

LINEAR/NONLINEAR SYSTEMS We have seen that a single differential equation can serve as a mathematical model for a single population in an environment. But if there are, say, two interacting and perhaps competing species living in the same environment (for example, rabbits and foxes), then a model for their populations $x(t)$

and $y(t)$ might be a system of two first-order differential equations such as

$$\begin{aligned}\frac{dx}{dt} &= g_1(t, x, y)\\ \frac{dy}{dt} &= g_2(t, x, y).\end{aligned} \tag{1}$$

When g_1 and g_2 are linear in the variables x and y—that is, g_1 and g_2 have the forms

$$g_1(t, x, y) = c_1x + c_2y + f_1(t) \quad \text{and} \quad g_2(t, x, y) = c_3x + c_4y + f_2(t),$$

where the coefficients c_i could depend on t—then (1) is said to be a **linear system.** A system of differential equations that is not linear is said to be **nonlinear.**

RADIOACTIVE SERIES In the discussion of radioactive decay in Sections 1.3 and 3.1 we assumed that the rate of decay was proportional to the number $A(t)$ of nuclei of the substance present at time t. When a substance decays by radioactivity, it usually doesn't just transmute in one step into a stable substance; rather, the first substance decays into another radioactive substance, which in turn decays into a third substance, and so on. This process, called a **radioactive decay series,** continues until a stable element is reached. For example, the uranium decay series is U-238 → Th-234 → · · · → Pb-206, where Pb-206 is a stable isotope of lead. The half-lives of the various elements in a radioactive series can range from billions of years (4.5×10^9 years for U-238) to a fraction of a second. Suppose a radioactive series is described schematically by $X \xrightarrow{-\lambda_1} Y \xrightarrow{-\lambda_2} Z$, where $k_1 = -\lambda_1 < 0$ and $k_2 = -\lambda_2 < 0$ are the decay constants for substances X and Y, respectively, and Z is a stable element. Suppose, too, that $x(t)$, $y(t)$, and $z(t)$ denote amounts of substances X, Y, and Z, respectively, remaining at time t. The decay of element X is described by

$$\frac{dx}{dt} = -\lambda_1 x,$$

whereas the rate at which the second element Y decays is the net rate

$$\frac{dy}{dt} = \lambda_1 x - \lambda_2 y,$$

since Y is *gaining* atoms from the decay of X and at the same time *losing* atoms because of its own decay. Since Z is a stable element, it is simply gaining atoms from the decay of element Y:

$$\frac{dz}{dt} = \lambda_2 y.$$

In other words, a model of the radioactive decay series for three elements is the linear system of three first-order differential equations

$$\begin{aligned}\frac{dx}{dt} &= -\lambda_1 x\\ \frac{dy}{dt} &= \lambda_1 x - \lambda_2 y\\ \frac{dz}{dt} &= \lambda_2 y.\end{aligned} \tag{2}$$

MIXTURES Consider the two tanks shown in Figure 3.3.1. Let us suppose for the sake of discussion that tank A contains 50 gallons of water in which 25 pounds of salt is dissolved. Suppose tank B contains 50 gallons of pure water. Liquid is pumped into and out of the tanks as indicated in the figure; the mixture exchanged between the two tanks and the liquid pumped out of tank B are assumed to be well stirred.

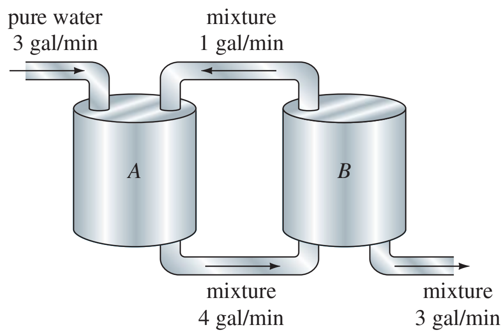

FIGURE 3.3.1 Connected mixing tanks

We wish to construct a mathematical model that describes the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks A and B, respectively, at time t.

By an analysis similar to that on page 23 in Section 1.3 and Example 5 of Section 3.1 we see that the net rate of change of $x_1(t)$ for tank A is

$$\frac{dx_1}{dt} = \overbrace{(3\text{ gal/min}) \cdot (0\text{ lb/gal}) + (1\text{ gal/min}) \cdot \left(\frac{x_2}{50}\text{ lb/gal}\right)}^{\text{input rate of salt}} - \overbrace{(4\text{ gal/min}) \cdot \left(\frac{x_1}{50}\text{ lb/gal}\right)}^{\text{output rate of salt}}$$

$$= -\frac{2}{25}x_1 + \frac{1}{50}x_2.$$

Similarly, for tank B the net rate of change of $x_2(t)$ is

$$\frac{dx_2}{dt} = 4 \cdot \frac{x_1}{50} - 3 \cdot \frac{x_2}{50} - 1 \cdot \frac{x_2}{50}$$

$$= \frac{2}{25}x_1 - \frac{2}{25}x_2.$$

Thus we obtain the linear system

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{2}{25}x_1 + \frac{1}{50}x_2 \\ \frac{dx_2}{dt} &= \frac{2}{25}x_1 - \frac{2}{25}x_2. \end{aligned} \tag{3}$$

Observe that the foregoing system is accompanied by the initial conditions $x_1(0) = 25$, $x_2(0) = 0$.

A PREDATOR-PREY MODEL Suppose that two different species of animals interact within the same environment or ecosystem, and suppose further that the first species eats only vegetation and the second eats only the first species. In other words, one species is a predator and the other is a prey. For example, wolves hunt grass-eating caribou, sharks devour little fish, and the snowy owl pursues an arctic rodent called the lemming. For the sake of discussion, let us imagine that the predators are foxes and the prey are rabbits.

Let $x(t)$ and $y(t)$ denote the fox and rabbit populations, respectively, at time t. If there were no rabbits, then one might expect that the foxes, lacking an adequate food supply, would decline in number according to

$$\frac{dx}{dt} = -ax, \qquad a > 0. \tag{4}$$

When rabbits are present in the environment, however, it seems reasonable that the number of encounters or interactions between these two species per unit time is jointly proportional to their populations x and y—that is, proportional to the product xy. Thus when rabbits are present, there is a supply of food, so foxes are added to the system at a rate bxy, $b > 0$. Adding this last rate to (4) gives a model for the fox population:

$$\frac{dx}{dt} = -ax + bxy. \tag{5}$$

On the other hand, if there were no foxes, then the rabbits would, with an added assumption of unlimited food supply, grow at a rate that is proportional to the number of rabbits present at time t:

$$\frac{dy}{dt} = dy, \qquad d > 0. \tag{6}$$

But when foxes are present, a model for the rabbit population is (6) decreased by cxy, $c > 0$—that is, decreased by the rate at which the rabbits are eaten during their encounters with the foxes:

$$\frac{dy}{dt} = dy - cxy. \tag{7}$$

Equations (5) and (7) constitute a system of nonlinear differential equations

$$\begin{aligned} \frac{dx}{dt} &= -ax + bxy = x(-a + by) \\ \frac{dy}{dt} &= dy - cxy = y(d - cx), \end{aligned} \tag{8}$$

where a, b, c, and d are positive constants. This famous system of equations is known as the **Lotka-Volterra predator-prey model.**

Except for two constant solutions, $x(t) = 0$, $y(t) = 0$ and $x(t) = d/c$, $y(t) = a/b$, the nonlinear system (8) cannot be solved in terms of elementary functions. However, we can analyze such systems quantitatively and qualitatively. See Chapter 9, "Numerical Solutions of Ordinary Differential Equations," and Chapter 10, "Plane Autonomous Systems."*

EXAMPLE 1 **Predator-Prey Model**

Suppose

$$\frac{dx}{dt} = -0.16x + 0.08xy$$

$$\frac{dy}{dt} = 4.5y - 0.9xy$$

represents a predator-prey model. Because we are dealing with populations, we have $x(t) \geq 0$, $y(t) \geq 0$. Figure 3.3.2, obtained with the aid of a numerical solver, shows typical population curves of the predators and prey for this model superimposed on the same coordinate axes. The initial conditions used were $x(0) = 4$, $y(0) = 4$. The curve in red represents the population $x(t)$ of the predators (foxes), and the blue curve is the population $y(t)$ of the prey (rabbits). Observe that the model seems to predict that both populations $x(t)$ and $y(t)$ are periodic in time. This makes intuitive sense because as the number of prey decreases, the predator population eventually decreases because of a diminished food supply; but attendant to a decrease in the number of predators is an increase in the number of prey; this in turn gives rise to an increased number of predators, which ultimately brings about another decrease in the number of prey. ■

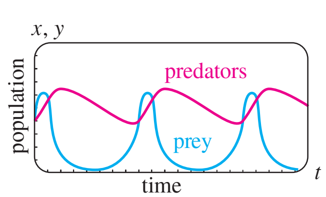

FIGURE 3.3.2 Populations of predators (red) and prey (blue) appear to be periodic

*Chapters 10–15 are in the expanded version of this text, *Differential Equations with Boundary-Value Problems.*

COMPETITION MODELS Now suppose two different species of animals occupy the same ecosystem, not as predator and prey but rather as competitors for the same resources (such as food and living space) in the system. In the absence of the other, let us assume that the rate at which each population grows is given by

$$\frac{dx}{dt} = ax \quad \text{and} \quad \frac{dy}{dt} = cy, \tag{9}$$

respectively.

Since the two species compete, another assumption might be that each of these rates is diminished simply by the influence, or existence, of the other population. Thus a model for the two populations is given by the linear system

$$\begin{aligned} \frac{dx}{dt} &= ax - by \\ \frac{dy}{dt} &= cy - dx, \end{aligned} \tag{10}$$

where a, b, c, and d are positive constants.

On the other hand, we might assume, as we did in (5), that each growth rate in (9) should be reduced by a rate proportional to the number of interactions between the two species:

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= cy - dxy. \end{aligned} \tag{11}$$

Inspection shows that this nonlinear system is similar to the Lotka-Volterra predator-prey model. Finally, it might be more realistic to replace the rates in (9), which indicate that the population of each species in isolation grows exponentially, with rates indicating that each population grows logistically (that is, over a long time the population is bounded):

$$\frac{dx}{dt} = a_1x - b_1x^2 \quad \text{and} \quad \frac{dy}{dt} = a_2y - b_2y^2. \tag{12}$$

When these new rates are decreased by rates proportional to the number of interactions, we obtain another nonlinear model:

$$\begin{aligned} \frac{dx}{dt} &= a_1x - b_1x^2 - c_1xy = x(a_1 - b_1x - c_1y) \\ \frac{dy}{dt} &= a_2y - b_2y^2 - c_2xy = y(a_2 - b_2y - c_2x), \end{aligned} \tag{13}$$

where all coefficients are positive. The linear system (10) and the nonlinear systems (11) and (13) are, of course, called **competition models.**

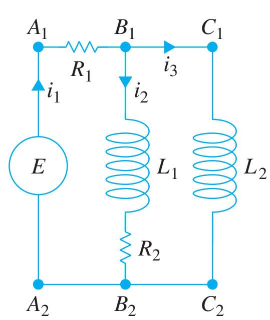

FIGURE 3.3.3 Network whose model is given in (17)

NETWORKS An electrical network having more than one loop also gives rise to simultaneous differential equations. As shown in Figure 3.3.3, the current $i_1(t)$ splits in the directions shown at point B_1, called a *branch point* of the network. By **Kirchhoff's first law** we can write

$$i_1(t) = i_2(t) + i_3(t). \tag{14}$$

We can also apply **Kirchhoff's second law** to each loop. For loop $A_1B_1B_2A_2A_1$, summing the voltage drops across each part of the loop gives

$$E(t) = i_1R_1 + L_1\frac{di_2}{dt} + i_2R_2. \tag{15}$$

Similarly, for loop $A_1B_1C_1C_2B_2A_2A_1$ we find

$$E(t) = i_1R_1 + L_2\frac{di_3}{dt}. \tag{16}$$

Using (14) to eliminate i_1 in (15) and (16) yields two linear first-order equations for the currents $i_2(t)$ and $i_3(t)$:

$$\begin{aligned} L_1\frac{di_2}{dt} + (R_1 + R_2)i_2 + R_1 i_3 &= E(t) \\ L_2\frac{di_3}{dt} + \qquad\quad R_1 i_2 + R_1 i_3 &= E(t). \end{aligned} \tag{17}$$

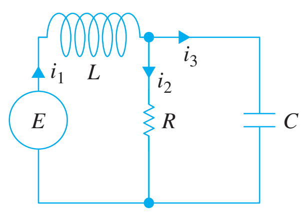

FIGURE 3.3.4 Network whose model is given in (18)

We leave it as an exercise (see Problem 14) to show that the system of differential equations describing the currents $i_1(t)$ and $i_2(t)$ in the network containing a resistor, an inductor, and a capacitor shown in Figure 3.3.4 is

$$\begin{aligned} L\frac{di_1}{dt} + Ri_2 \qquad &= E(t) \\ RC\frac{di_2}{dt} + \quad i_2 - i_1 &= 0. \end{aligned} \tag{18}$$

EXERCISES 3.3

Answers to selected odd-numbered problems begin on page ANS-4.

Radioactive Series

1. We have not discussed methods by which systems of first-order differential equations can be solved. Nevertheless, systems such as (2) can be solved with no knowledge other than how to solve a single linear first-order equation. Find a solution of (2) subject to the initial conditions $x(0) = x_0$, $y(0) = 0$, $z(0) = 0$.

2. In Problem 1 suppose that time is measured in days, that the decay constants are $k_1 = -0.138629$ and $k_2 = -0.004951$, and that $x_0 = 20$. Use a graphing utility to obtain the graphs of the solutions $x(t)$, $y(t)$, and $z(t)$ on the same set of coordinate axes. Use the graphs to approximate the half-lives of substances X and Y.

3. Use the graphs in Problem 2 to approximate the times when the amounts $x(t)$ and $y(t)$ are the same, the times when the amounts $x(t)$ and $z(t)$ are the same, and the times when the amounts $y(t)$ and $z(t)$ are the same. Why does the time that is determined when the amounts $y(t)$ and $z(t)$ are the same make intuitive sense?

4. Construct a mathematical model for a radioactive series of four elements W, X, Y, and Z, where Z is a stable element.

Mixtures

5. Consider two tanks A and B, with liquid being pumped in and out at the same rates, as described by the system of equations (3). What is the system of differential equations if, instead of pure water, a brine solution containing 2 pounds of salt per gallon is pumped into tank A?

6. Use the information given in Figure 3.3.5 to construct a mathematical model for the number of pounds of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ at time t in tanks A, B, and C, respectively.

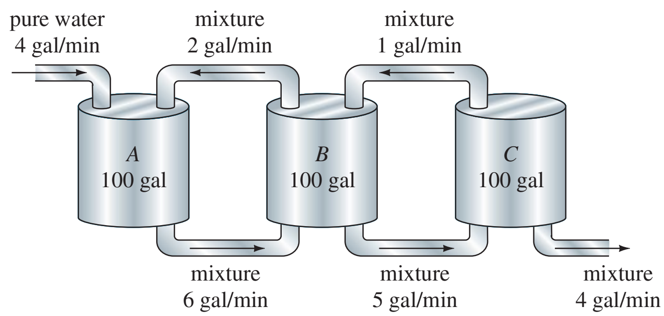

FIGURE 3.3.5 Mixing tanks in Problem 6

7. Two very large tanks A and B are each partially filled with 100 gallons of brine. Initially, 100 pounds of salt is dissolved in the solution in tank A and 50 pounds of salt is dissolved in the solution in tank B. The system is closed in that the well-stirred liquid is pumped only between the tanks, as shown in Figure 3.3.6.

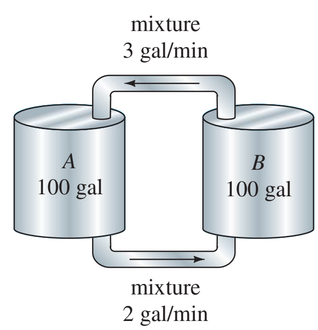

FIGURE 3.3.6 Mixing tanks in Problem 7

(a) Use the information given in the figure to construct a mathematical model for the number of pounds of salt $x_1(t)$ and $x_2(t)$ at time t in tanks A and B, respectively.